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Index for Finite Real Factors

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Abstract

For real factors the notions of the coupling constant and the index are introduced and investigated. The possible values of the index for type II_1 real factors are calculated, in a similar way as for the complex case.

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1 Introduction

In 1930's Von Neumann and Murray introduced the notion of coupling constant for finite factors (see [14, 15, 16]). In 1983, V.Jones suggested a new approach to this notion, defined the notion of index for type II_1 factors, and proved a surprising theorem on values of the index for subfactors (see [6]). He also introduced a very important technique in the proof of this theorem: the towers of algebras. Since then this theory has become a focus of many fields in mathematics and physics ([7]). In [8], H.Kosaki extended the notion of the index to an arbitrary (normal faithful) expectation from a factor onto a subfactor. While Jones' definition of the index is based on the coupling constant, Kosaki's definition of the index of an expectation relies on the notion of spatial derivatives due to A.Connes [2] as well as the theory of operator-value weights due to U.Haagerup [5]. In [8, 9], it was shown that many fundamental properties of the Jones index in the type II_1 case can be extended to the general setting. At the present time, the theory of index thanks to works by V.Jones, P.Loi, R.Longo, H.Kosaki and other mathematicians is deeply developed and has many applications in the theory of operator algebras and physics (see also [12, 13]).

Unlike to the complex case for real factors the notion of coupling constant (therefore the notion of index as well) has not been investigated. In the present paper the notions of the real coupling constant and the index for finite real factors are introduced and investigated. The main tool in our approach is the reduction of real factors to involutive *-anti-automorphisms of their complex enveloping von Neumann algebras.

2 Preliminaries

Let $B(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H. A weakly closed *-subalgebra $\mathfrak{A}$ containing the identity operator $\mathbb{1}$ in $B(H)$ is called a W*-*algebra*. A real *-subalgebra $\Re \subset B(H)$ is called a *real* W*-*algebra* if it is closed in the weak operator topology and $\Re \cap i\Re = \{0\}$. A real W*-algebra $\Re$ is called a *real factor* if its center $Z(\Re)$ consists of the elements $\{\lambda \mathbb{1}, \lambda \in \mathbb{R}\}$. We say that a real W*-algebra $\Re$ is of the type I_{fin}, I_∞, II_1, II_∞, or III_λ, $(0 \leq \lambda \leq 1)$ if the enveloping W*-algebra

$\mathfrak{A}(\mathfrak{R})$ has the corresponding type in the ordinary classification of W*-algebras. A linear mapping α of an algebra into itself with $\alpha(x^*) = \alpha(x)^*$ is called an *-*automorphism* if $\alpha(xy) = \alpha(x)\alpha(y)$; it is called an *involutive *-antiautomorphism* if $\alpha(xy) = \alpha(y)\alpha(x)$ and $\alpha^2(x) = x$. If α is an involutive *-antiautomorphism of a W*-algebra M, we denote by (M, α) the real W*-algebra generated by α, i.e. $(M, \alpha) = \{x \in M : \alpha(x) = x^*\}$. Conversely, every real W*-algebra $\mathfrak{R}$ is of the form (M, α), where M is the complex envelope of $\mathfrak{R}$ and α is an involutive *-antiautomorphism of M (see [1,4,17]). Therefore we shall identify from now on the real von Neumann algebra $\mathfrak{R}$ with the pair (M, α).

3 Canonical representation

Let M $(\subset B(H))$ be a finite factor and let τ be the unique faithful normal tracial state of M. If α is an involutive *-antiautomorphism of M, then it is clear that τ is automatically α-invariant. Denote by $L^2(M)$ the completion of M with respect to the norm $\|x\|_2 = \tau(x^*x)^{1/2}$. Similarly by $L^2(M, \alpha)$ we denote the completion of the real factor (M, α). Then it is obvious that the Hilbert space $L^2(M)$ and the algebra $B(L^2(M))$ of all bounded linear operators on it are the complexifications of the real Hilbert space $L^2(M, \alpha)$ and of $B_r(L^2(M, \alpha))$, respectively, where $B_r(L^2(M, \alpha))$ is the algebra of all bounded linear operators on the real Hilbert space $L^2(M, \alpha)$. Moreover, it is easy to show that the Hilbert spaces $L^2(M, \alpha)$ and $L^2(M)$ are separable.

For each $x \in M$, set $\lambda(x)y = xy$, for all $y \in M$. Clearly, $\|\lambda(x)y\|_2 \leq \|x\|\|y\|_2$. Thus λ can be uniquely extended to a bounded linear operator on $L^2(M)$, still denoted by $\lambda(x)$. Then we obtain a faithful W*-representation $(\lambda, L^2(M))$ of M. In a similar way, taking the map λ_r defined as $\lambda_r(x)y = xy$ (for all $x, y \in (M, \alpha)$) we obtain a faithful real $*$-representation $(\lambda_r, L^2(M, \alpha))$ of (M, α).

Theorem 3.1 *The map $\beta : \lambda(M) \to \lambda(M)$ defined as $\beta(\lambda_x) = \lambda_{\alpha(x)}$ is an involutive *-antiautomorphism of $\lambda(M)$. Moreover, β and α are also related in the following way: $(M, \alpha)_\beta = \lambda_r(M, \alpha)$, where $(M, \alpha)_\beta = \{\lambda_x \in \lambda(M) : \beta(\lambda_x) = \lambda_x^*\}$ is the real W*-algebra, generated by β, i.e. $(M, \alpha)_\beta = (\lambda(M), \beta)$.*

Proof. The first part of the assertion is trivial. Further, let $\lambda_x \in (M, \alpha)_\beta$. Since

$\beta(\lambda_x) = \lambda_x^*$, then $\lambda_{\alpha(x)} = \lambda_{x^*}$. Hence $\alpha(x) = x^*$, i.e. $x \in (M, \alpha)$. Then from

$$\lambda_x \in \lambda(M) \subset B(L^2(M)) = B_r(L^2(M, \alpha)) + iB_r(L^2(M, \alpha))$$

we have $(M, \alpha)_\beta \subset B_r(L^2(M, \alpha))$. Hence $(M, \alpha)_\beta \subset \lambda_r(M, \alpha)$, since $\lambda_r(M, \alpha) = \{\lambda_x^r \in B_r(L^2(M, \alpha)) :$ for $\alpha(x) = x^*$ and $\alpha(y) = y^*$, $\lambda_x^r(y) := xy\}$.

Now let $\lambda_x^r \in \lambda_r(M, \alpha)$. Then $\alpha(x) = x^*$ and $\lambda_x^r \in \lambda_r(M, \alpha) \subset \lambda(M)$. Hence $\beta(\lambda_x^r) = \lambda_{\alpha(x)}^r = \lambda_{x^*}^r = (\lambda_x^r)^*$, therefore $\lambda_x^r \in (M, \alpha)_\beta$. $\square$

Corollary 3.2 *$\lambda_r(M, \alpha)$ is a real W*-algebra, and $\lambda(M)$ is the complexification of $\lambda_r(M, \alpha)$, i.e. $\lambda_r(M, \alpha) + i\lambda_r(M, \alpha) = \lambda(M)$. Moreover, $\{\lambda_r, L^2(M, \alpha)\}$ is a faithful real W*-representation of (M, α).*

This representation will be called the canonical W*-representation of (M, α).

4 Commutant of the canonical representation

Since $\|x\|_2 = \|x^*\|_2$ for all $x \in M$, the map $J : x \to x^*$ can be uniquely extended to a conjugate linear isometry on $L^2(M)$, still denoted by J. From the theory of W*-algebras it is well-known that $\lambda(M)' = J\lambda(M)J$ and $\lambda(M) = J\lambda(M)'J$. Similarly to Theorem 3.1 and Corollary 3.2 we have the following assertion

Theorem 4.1 *The map $\beta' : \lambda(M)' \to \lambda(M)'$ defined as $\beta'(\cdot) = J\beta(J \cdot J)J$, is an involutive *-antiautomorphism of $\lambda(M)'$. The set $\lambda_r(M, \alpha)' = \{\lambda_{x'} \in \lambda(M)' : \beta'(\lambda_{x'}) = \lambda_{x'}^*\}$ is a real W*-algebra, and $\lambda(M)'$ is the complexification of $\lambda_r(M, \alpha)'$, i.e. $\lambda_r(M, \alpha)' + i\lambda_r(M, \alpha)' = \lambda(M)'$.*

We have the following connection between $\lambda_r(M, \alpha)$ and $\lambda_r(M, \alpha)'$.

Theorem 4.2 $\lambda_r(M, \alpha)' = J\lambda_r(M, \alpha)J$.

Proof. Since $\lambda_x \in \lambda_r(M, \alpha)$ implies that $J\lambda_x J \in J\lambda_r(M, \alpha)J$ and $\beta(\lambda_x) = \lambda_x^*$, we have

$$\beta'(J\lambda_x J) = J\beta(JJ\lambda_x JJ)J = J\beta(\lambda_x)J = J\lambda_x^* J = (J\lambda_x J)^*.$$

Hence $J\lambda_x J \in \lambda_r(M,\alpha)'$, i.e. $J\lambda_r(M,\alpha)J \subset \lambda_r(M,\alpha)'$.

Conversely, let $\lambda_{x'} \in \lambda_r(M,\alpha)' \subset \lambda(M)' = J\lambda(M)J$. Then $\lambda_{x'} = J\lambda_y J$, for some $\lambda_y \in \lambda(M)$. Since $\beta(\lambda_{x'}) = \lambda_{x'}^*$, we have $\beta'(J\lambda_y J) = J\lambda_y^* J$, i.e. $J\beta(JJ\lambda_y JJ)J = J\lambda_y^* J$. Hence $J^2\beta(\lambda_y)J^2 = J^2\lambda_y^* J^2$, i.e. $\beta(\lambda_y) = \lambda_y^*$. Therefore $\lambda_y \in \lambda_r(M,\alpha)$. Thus we obtain $\lambda_{x'} = J\lambda_y J = J\lambda_r(M,\alpha)J$, and therefore $\lambda_r(M,\alpha)' \subset J\lambda_r(M,\alpha)J$. $\square$

Theorem 4.3 *The real W*-algebra $\lambda_r(M,\alpha)'$ is the commutant of $\lambda_r(M,\alpha)$ in the algebra $B_r(L^2(M,\alpha))$, i.e. $\lambda_r(M,\alpha)' = \{\lambda_x \in B_r(L^2(M,\alpha)) : \lambda_x\lambda_y = \lambda_y\lambda_x, \ \forall\ \lambda_y \in \lambda_r(M,\alpha)\}$*

Proof. Similarly to the proof of Theorem 3.1 for $\beta'(\lambda_x) = \lambda_x^*$ we have $\lambda_x \in B_r(L^2(M,\alpha))$. Therefore $\lambda_r(M,\alpha)' \subset B_r(L^2(M,\alpha))$. On the other hand for any $\lambda_x \in \lambda_r(M,\alpha)' \subset \lambda(M)'$ and $\lambda_y \in \lambda_r(M,\alpha) \subset \lambda(M)$, we have $\lambda_x\lambda_y = \lambda_y\lambda_x$. $\square$

5 Relations between faithful nondegenerate W*-representations and the canonical representation

Theorem 5.1 *Let $M_1 \subset B(H_1)$ and $M_2 \subset B(H_2)$ be two W^*-algebras and let α_i be an involutive *-antiautomorphism of M_i, $i = 1, 2$. If $\Phi : M_1 \to M_2$ is a normal *-homomorphism with $\Phi \circ \alpha_1 = \alpha_2 \circ \Phi$, then*

$$\Phi = \Phi_3 \circ \Phi_2 \circ \Phi_1,$$

where

*Φ_1 is a *-homomorphism from M_1 onto $M_1\overline{\otimes}\, \mathbb{C}\mathbb{1}_L$ with $\Phi_1 \circ \alpha_1 = \tilde{\alpha_1} \circ \Phi_1$ defined as $\Phi(a) = a \otimes \mathbb{1}_L$, where $\mathbb{1}_L$ is the identity operator on an appropriate Hilbert space L and $\tilde{\alpha_1} = \alpha_1 \otimes id$;*

*Φ_2 is a *-homomorphism from $M_1\overline{\otimes}\, \mathbb{C}\mathbb{1}_L$ onto $(M_1\overline{\otimes}\, \mathbb{C}\mathbb{1}_L)p'$ with $\Phi_2 \circ \tilde{\alpha_1} = \overline{\alpha}_1 \circ \Phi_2$ defined as $\Phi_2(a \otimes \mathbb{1}_L) = (a \otimes \mathbb{1}_L)p'$, where p' is a projection from $(M_1\overline{\otimes}\, \mathbb{C}\mathbb{1}_L)'$ with $\tilde{\alpha_1}'(p') = p'$ and $\tilde{\alpha_1}' = J_1\tilde{\alpha_1}(J_1(.)J_1)J_1 \otimes id$, $\overline{\alpha}_1(\cdot\ p') = \tilde{\alpha_1}(\cdot)p'$;*

*Φ_3 is a *-isomorphism from $(M_1\overline{\otimes}\, \mathbb{C}\mathbb{1}_L)p'$ to M_2 with $\Phi_3 \circ \tilde{\alpha_1} = \alpha_2 \circ \Phi_3$.*

Proof. First we assume that (M_2,α_2) admits a cyclic vector η. In this case $\overline{(M_2,\alpha_2)\eta} = H_2^r$ is a real Hilbert space and

$$\overline{M_2\eta} = \overline{(M_2,\alpha_2)\eta} + i\overline{(M_2,\alpha_2)\eta} = H_2^r + iH_2^r = H_2,$$

hence η is a cyclic vector of M_2. Since $\Phi \circ \alpha_1 = \alpha_2 \circ \Phi$, for all $a \in (M_1,\alpha_1)$ we have $\alpha_2\big(\Phi(a)\big) = \Phi\big(\alpha_1(a)\big) = \Phi(a^*) = \Phi(a)^*$, i.e. $\Phi(a) \in (M_2,\alpha_2)$. Hence $\Phi\big((M_1,\alpha_1)\big) \subset (M_2,\alpha_2)$. Define a functional φ by

$$\varphi(a) =< \Phi(a)\eta,\eta >, \qquad a \in (M_1,\alpha_1).$$

Obviously, φ is a normal positive functional on (M_1,α_1). We can extend φ by linearity to a functional on M_1 (still denoted by φ) such that

$$\varphi(a+ib) = \varphi(a) + i\varphi(b), \qquad a,b \in (M_1,\alpha_1),$$

which clearly also is a normal positive functional. Let H_1^r be a real Hilbert space with $H_1^r + iH_1^r = H_1$ such that $(M_1,\alpha_1) \subset B(H_1^r)$. By [11, 4.2.1] there is a sequence $(\xi_n) \subset H_1^r$ with $\sum\limits_n \|\xi_n\|^2 < \infty$ such that $\varphi(a) = \sum\limits_n < a\xi_n,\xi_n >$, for all $a \in (M_1,\alpha_1)$. Set $L_r = \ell_2^r = \{(x_n) \subset \mathbb{R} : \sum_n x_n^2 < \infty\}$, $L = L_r + iL_r$, $\xi = (\xi_n) \subset H_1^r \otimes L_r$ and $\Phi_1(a) = a \otimes \mathbb{1}_L$ for all $a \in M_1$. Then Φ_1 is a map from M_1 to $M_1\overline{\otimes}\, \mathbb{C}\mathbb{1}_L$ and

$$\begin{aligned}(\Phi_1 \circ \alpha_1)(a) &= \Phi_1(\alpha_1(a)) = \alpha_1(a) \otimes \mathbb{1}_L = (\alpha_1 \otimes id)(a \otimes \mathbb{1}_L)\\ &= \tilde{\alpha_1}(\Phi_1(a)) = (\tilde{\alpha_1} \circ \Phi_1)(a),\end{aligned}$$

i.e. $\Phi_1 \circ \alpha_1 = \tilde{\alpha_1} \circ \Phi_1$. Moreover, for all $a \in (M_1,\alpha_1)$ we have

$$< \Phi_1(a)\xi,\xi >=< (a \otimes \mathbb{1}_{L_r})\xi,\xi >= \sum_n < a\xi_n,\xi_n >= \varphi(a).$$

Let p' be the projection from $H_1^r \otimes L_r$ to $\overline{\Phi_1\big((M_1,\alpha_1)\big)\xi}$. Then for all $x = a \otimes \mathbb{1}_{L_r} \in ((M_1,\alpha_1)\overline{\otimes}\, \mathbb{R}\mathbb{1}_{L_r})$ we have

$$\begin{aligned}(p'x)\xi &= p'((a \otimes \mathbb{1}_{L_r})\xi) = p'(\Phi_1(a)\xi) = \Phi_1(a)\xi\\ &= (a \otimes \mathbb{1}_{L_r})\xi = x\xi = x((\mathbb{1} \otimes \mathbb{1}_{L_r})\xi) = x(\Phi_1(\mathbb{1})\xi)\\ &= x(p'(\Phi_1(\mathbb{1})\xi)) = x(p'(\xi)) = (xp')\xi.\end{aligned}$$

Similarly, for all $\gamma \in H_1^r \otimes L_r$ with $\gamma \neq \xi$ we also obtain

$$\begin{aligned}(p'x)\gamma &= p'(\Phi_1(a)\gamma) = \theta = x(\theta) = x(p'(\Phi_1(\mathbb{1})\gamma)) \\ &= xp'((\mathbb{1} \otimes \mathbb{1}_{L_r})\gamma) = xp'(\gamma).\end{aligned}$$

Therefore $p'x = xp'$, i.e. $p' \in ((M_1, \alpha_1)\overline{\otimes}\ \mathbb{R}\mathbb{1}_{L_r})'$. Hence $p' \in (M_1\overline{\otimes}\ \mathbb{C}\mathbb{1}_L)'$ and for $\tilde{\alpha_1}' = J_1\tilde{\alpha_1}(J_1(.)J_1)J_1 \otimes id$ we have $\tilde{\alpha_1}'(p') = p'$.

Define the map $\Phi_2 : M_1\overline{\otimes}\ \mathbb{C}\mathbb{1}_L \to (M_1\overline{\otimes}\ \mathbb{C}\mathbb{1}_L)p'$ as $\Phi_2(a \otimes \mathbb{1}_L) = (a \otimes \mathbb{1}_L)p'$, $a \in M_1$. Then

$$\begin{aligned}(\Phi_2 \circ \tilde{\alpha_1})(a \otimes \mathbb{1}_L) &= \Phi_2(\tilde{\alpha_1}(a \otimes \mathbb{1}_L)) = \Phi_2(\alpha_1(a) \otimes \mathbb{1}_L) \\ &= (\alpha_1(a) \otimes \mathbb{1}_L)p' = \tilde{\alpha_1}(a \otimes \mathbb{1}_L)p' \\ &= \overline{\alpha}_1\big((a \otimes \mathbb{1}_L)p'\big) = \overline{\alpha}_1(\Phi_2(a \otimes \mathbb{1}_L)) \\ &= (\overline{\alpha}_1 \circ \Phi_2)(a \otimes \mathbb{1}_L),\end{aligned}$$

hence $\Phi_2 \circ \tilde{\alpha_1} = \overline{\alpha}_1 \circ \Phi_2$. Since $p'\xi = p'((\mathbb{1} \otimes \mathbb{1}_L)\xi) = p'(\Phi_1(\mathbb{1})\xi) = \Phi_1(\mathbb{1})\xi = \xi$, we have

$$\begin{aligned}< (\Phi_2 \circ \Phi_1)(a)\xi, \xi > &= < (\Phi_2(a \otimes \mathbb{1}_{L_r}))\xi, \xi >=< (a \otimes \mathbb{1}_{L_r})p'\xi, \xi > \\ &= < (a \otimes \mathbb{1}_{L_r})\xi, \xi >=< \Phi_1(a)\xi, \xi >= \varphi(a),\end{aligned}$$

for all $a \in (M_1, \alpha_1)$, i.e. $\varphi(a) =< (\Phi_2 \circ \Phi_1)(a)\xi, \xi >$.

Now, define a linear map $u : \Phi\big((M_1, \alpha_1)\big)\eta \to p'(H_1^r \otimes L_r)$ as follows:

$$u\Phi(a)\eta = (\Phi_2 \circ \Phi_1)(a)\xi = p'(a\xi_n) = (a\xi_n) \quad (a \in (M_1, \alpha_1)).$$

Since $u\Phi(a)\eta = (\Phi_2 \circ \Phi_1)(a)\xi$ and $< \Phi(a)\eta, \eta >= \varphi(a) =< (\Phi_2 \circ \Phi_1)(a)\xi, \xi >$ $(a \in (M_1, \alpha_1))$, it follows that $\|u\Phi(a)\eta\|' = \|\Phi(a)\eta\|_2^r$, i.e. the map u is an isometry, where $\|\cdot\|_2^r$ is the norm of the space H_2 and $\|\cdot\|'$ is the norm of the space $H_1^r \otimes L_r$. Moreover, since $\Phi\big((M_1, \alpha_1)\big)\eta = (M_2, \alpha_2)\eta$, $(\Phi_2 \circ \Phi_1)\big((M_1, \alpha_1)\big)\xi = \Phi_1\big((M_1, \alpha_1)\big)\xi$, and

$$\overline{\Phi\big((M_1, \alpha_1)\big)\eta} = \overline{(M_2, \alpha_2)\eta} = H_2^r\ ,$$

$$\overline{(\Phi_2 \circ \Phi_1)\big((M_1, \alpha_1)\big)\xi} = \overline{\Phi_1\big((M_1, \alpha_1)\big)\xi} = p'(H_1^r \otimes L_r),$$

u can be extended to a unitary operator $\overline{u} : H_2^r \to p'(H_1^r \otimes L)$. Clearly,

$$\overline{u}\Phi(a)\overline{u}^{\,-1} = \Phi_2 \circ \Phi_1(a), \quad a \in (M_1, \alpha_1). \tag{5.1}$$

Therefore we can define a spatial real *-isomorphism $\Phi_3 : \big((M_1, \alpha_1)\overline{\otimes}\ \mathbb{R}\mathbb{1}_{L_r}\big)p' \to (M_2, \alpha_2)$ as $\Phi_3(.) = \overline{u}^{\,-1}(.)\overline{u}$, and it can be extended to a spatial *-isomorphism (still denoted by Φ_3) $\Phi_3 : (M_1 \overline{\otimes}\ \mathbb{C}\mathbb{1}_L)p' \to M_2$ as $\Phi_3(a + ib) = \Phi_3(a) + i\Phi_3(b)$, where $a, b \in \big((M_1, \alpha_1)\overline{\otimes}\ \mathbb{R}\mathbb{1}_{L_r}\big)p'$. Then, by (5.1) we have $\Phi = \Phi_3 \circ \Phi_2 \circ \Phi_1$.

Considering now the general case, the real Hilbert space H_2^r with $H_2^r + iH_2^r = H_2$ can be decomposed as

$$H_2^r = \oplus_l H_2^l \quad \text{and} \quad H_2^l = \overline{(M_2, \alpha_2)\eta_l}, \quad \text{where} \quad \eta_l \in H_2^r, \quad \text{for } l \in \mathbb{N}.$$

Let $q_l' : H_2^r \to \overline{(M_2, \alpha_2)\eta_l} = H_2^l$ be the projection. Then $q_l' \in (M_2, \alpha_2)'$, for all l. For each l, $\Phi_l = q_l'\Phi : (M_1, \alpha_1) \to (M_2, \alpha_2)q_l'$ is a normal *-homomorphism, which can be extended to a normal *-homomorphism $\Phi_l : M_1 \to M_2 q_l'$. Then, by the above argument $\Phi_l = \Phi_3^{(l)} \circ \Phi_2^{(l)} \circ \Phi_1^{(l)}$, for all l. Set $\Phi_i = \oplus_l \Phi_i^{(l)}$, $i = 1, 2, 3$. Then $\Phi = \Phi_3 \circ \Phi_2 \circ \Phi_1$ and the maps Φ_3, Φ_2, Φ_1 satisfy all our conditions. $\square$

Theorem 5.2 *Let M be a finite factor and let α be an involutive *-antiautomorphism of M. If $\{\pi, H\}$ is a faithful nondegenerate W*-representation of M and $\pi \circ \alpha = \tilde{\alpha} \circ \pi$ for an involutive *-antiautomorphism $\tilde{\alpha}$ of $\pi(M)$, then there exist a projection $p' \in (\lambda_r(M, \alpha) \otimes \mathbb{1}_{K_r})'$, and a unitary operator $u : H_r \to p'(L^2(M, \alpha) \otimes K_r)$ such that*

$$u\pi(x) = (\lambda(x) \otimes \mathbb{1}_K)u, \qquad x \in M,$$

i.e., the real W-algebras $\pi(M, \alpha)$ $(= (\pi(M), \tilde{\alpha}))$ and $(\lambda_r(M, \alpha) \otimes \mathbb{1}_{K_r})p'$ are spatially *-isomorphic and therefore the W*-algebras $\pi(M)$ and $(\lambda(M) \otimes \mathbb{1}_K)p'$ are also spatially *-isomorphic; where K_r is a separable infinite dimensional Hilbert space, and $K = K_r + iK_r$.*

Proof. Set $M_1 = \lambda(M)$ and $M_2 = \pi(M)$. Define the map $\Phi : M_1 \to M_2$ by $\Phi(\lambda(x)) = \pi(x)$. Then Φ is a *-isomorphism and $\Phi\big(\lambda_r(M, \alpha)\big) \subset (\pi(M), \tilde{\alpha})$. Now the

conclusion follows immediately from Theorem 5.1 and the separability of H. $\square$

6 The coupling constants for real factors

If M $(\subset B(H))$ is a finite factor with the finite commutant M', the *coupling constant* $\dim_M(H)$ of M is defined as $\mathrm{tr}_M(E_\xi^{M'})/\mathrm{tr}_{M'}(E_\xi^M)$, where ξ is a non-zero vector in H, tr_A denotes the normalized trace and E_ξ^A is the projection onto the closure of the subspace $A\xi$. This definition, due to Murray and von Neumann in [14], is independent of ξ. We recall some properties of the coupling constant ([14, 15], see also [6], [3], [10, Ch. 17])

(6.1a) $\dim_M(L^2(M)) = 1$,

(6.1b) $\dim_M(H) \cdot \dim_{M'}(H) = 1$,

(6.1c) If $\{\pi, H\}$ and $\{\pi', H'\}$ are faithful nondegenerate W^*representations of M, then $\dim_M(H) = \dim_M(H')$ if and only if $\{\pi, H\} \cong \{\pi', H'\}$, i.e. if these $\mathrm{W}^* -$ representations are spatially $* -$isomorphic.

(6.1d) If $\{\pi_i, H_i\}_{i\geq 1}$ is a sequence of faithful nondegenerate W^*representations of M, then $\dim_M\big(\sum_i H_i\big) = \sum_i \dim_M(H_i)$,

(6.1e) If $\{\pi, H\}$ is a faithful nondegenerate W^*representation of M, then $\pi(M)'$ is finite if and only if $\dim_M(H) < \infty$.

(6.1f) $\dim_M(H) \geq 1$ (resp. ≤ 1) if and only if M admits a separating (resp. cyclic) vector.

We are now in a position to give the definition of the coupling constant for real finite factors. Let us first prove an auxiliary Lemma.

Lemma 6.1 *If H is a real Hilbert space and $R \subset B(H)$ is a real W^*-algebra, then $R' + iR' = (R + iR)'$, where the latter commutant is taken in $B(H + iH)$.*

Proof. A straightforward calculation shows that $R' + iR' \subset (R + iR)'$. Since $B(H+iH) = B(H)+iB(H)$ (see [11, Proposition 1.1.11]), for each $a' \in (R+iR)'$ there

exist $x', y' \in B(H)$ such that $a' = x' + iy'$. Since $a'b = ba'$ for all $b = x + iy \in R + iR$, we have that $x'x - y'y = xx' - yy'$ and $x'x + y'y = xx' + yy'$. Hence $x'x = xx'$ and $y'y = yy'$, i.e. $x', y' \in R'$. Therefore $a' \in R' + iR'$. $\square$

Now, let M be a finite factor and let α be an involutive *-antiautomorphism of M. If $\{\pi, H\}$ is a faithful nondegenerate W*-representation of M, and $\pi \circ \alpha = \tilde{\alpha} \circ \pi$ for an involutive *-antiautomorphism $\tilde{\alpha}$ of $\pi(M)$, then by Lemma 6.1 we have $(\pi(M), \tilde{\alpha})' + i(\pi(M), \tilde{\alpha})' = \pi(M)'$. Since the von Neumann algebra $\pi(M)'$ is semi-finite, the real factor $(\pi(M), \tilde{\alpha})'$ is also semi-finite. Thus there exists a unique (up to multiplication by a positive constant) faithful normal semi-finite $\tilde{\alpha}$-invariant trace on $\pi(M)'_+$. We define a *natural $\tilde{\alpha}$-invariant trace* on $\pi(M)'_+$ as follows.

(i) If $\{\pi, H\} = \{\lambda \otimes \mathbb{1}, L^2(M) \otimes K\}$, where K is a countably infinite dimensional Hilbert space, then the von Neumann algebra $(\lambda(M) \otimes \mathbb{1}_K)' = J\lambda(M)J\overline{\otimes}B(K)$ is infinite and for the real factor (M, α) we have

$$\{\pi|_{(M,\alpha)}, H_r\} = \{\lambda_r \otimes \mathbb{1}, L^2(M, \alpha) \otimes K_r\},$$

$$(\lambda_r(M, \alpha) \otimes \mathbb{1}_{K_r})' = J\lambda_r(M, \alpha)J\overline{\otimes}B(K_r)$$

(further, for the sake of convenience, we shall write π, instead of $\pi|_{(M,\alpha)}$). Pick an orthogonal normalized basis $\{e_i\}_{i\in\Lambda}$ of K, where $|\Lambda| = \dim_{\mathbb{C}} K$. Then each element $t' \in (\lambda(M) \otimes \mathbb{1}_K)'$ can be uniquely represented as $t' = \big(J\lambda(x_{ij})J\big)$, where $x_{ij} \in M$, for all i, j. If $t' \in (\lambda_r(M, \alpha) \otimes \mathbb{1}_{K_r})'$, i.e. $\tilde{\alpha}'(t') = (t')^*$, then it is not difficult to show that $x_{ij} \in (M, \alpha)$ (i.e. $\alpha(x_{ij}) = x_{ij}^*$), for all i, j. Define the natural trace as follows (see [10, 17.1.4 (i)]):

$$\mathtt{Tr}'_{L^2(M)\otimes K}(t') = \sum_{i\in\Lambda} \tau(x_{ii}), \qquad t' = \big(J\lambda(x_{ij})J\big) \in (\lambda(M) \otimes \mathbb{1}_K)'_+,$$

where τ is the unique faithful normal (and hence α-invariant) tracial state on M. It is easy to show that the definition of $\mathtt{Tr}'_{L^2(M)\otimes K}$ is independent of the choice of $\{e_i\}$ and $\mathtt{Tr}'_{L^2(M)\otimes K}$ is a faithful semi-finite normal trace on $(\lambda(M)\otimes \mathbb{1}_K)'_+$. Moreover, since

$$(\mathtt{Tr}'_{L^2(M)\otimes K}\circ\tilde{\alpha}')(t') = \mathtt{Tr}'_{L^2(M)\otimes K}(\tilde{\alpha}'(t')) = \mathtt{Tr}'_{L^2(M)\otimes K}((t')^*)$$

$$= \sum_i \tau(x_{ii}^*) = \sum_i \tau(\alpha(x_{ii})) = \sum_i (\tau\circ\alpha)(x_{ii}) = \sum_i \tau(x_{ii})$$

$$= \mathtt{Tr}'_{L^2(M)\otimes K}(t'),$$

we have that $\mathtt{Tr}'_{L^2(M)\otimes K}$ is $\tilde{\alpha}'$-invariant. Therefore, for $\mathtt{Tr}'_{L^2(M,\alpha)\otimes K_r}$ defined as follows

$$\mathtt{Tr}'_{L^2(M,\alpha)\otimes K_r}(t') = \sum_i \tau(x_{ii}), \;\; t' = \big(J\lambda(x_{ij})J\big) \in (\lambda_r(M,\alpha)\otimes \mathbb{1}_{K_r})'_+,$$

we have

$$\mathtt{Tr}'_{L^2(M)\otimes K}\Big|_{(\lambda_r(M,\alpha)\otimes\mathbb{1}_{K_r})'} = \mathtt{Tr}'_{L^2(M,\alpha)\otimes K_r} .$$

(ii) For a general faithful nondegenerate W*-representation $\{\pi, H\}$ of M with $\pi\circ\alpha = \tilde{\alpha}\circ\pi$ by Theorem 5.2 there are a projection $p' \in (\lambda_r(M,\alpha)\otimes\mathbb{1}_{K_r})'$ and a unitary $u : H_r \to p'(L^2(M,\alpha)\otimes K_r)$ such that

$$u\pi(x)u^* = (\lambda(x)\otimes\mathbb{1}_K)p', \qquad x\in M,$$

where K_r is a real Hilbert space and $K = K_r + iK_r$. Then we define the natural trace as follows (see [10, 17.1.4 (ii)]):

$$\mathtt{Tr}'_H(t') = \mathtt{Tr}'_{L^2(M)\otimes K}(ut'u^*), \qquad t'\in\pi(M)'_+ .$$

The definition of $\mathtt{Tr}'_H$ is independent on the choice of u and p', and $\mathtt{Tr}'_H$ is a faithful normal trace on $\pi(M)'_+$. Since $\tilde{\alpha}'(u) = u^*$, we have that $\mathtt{Tr}'_H$ is $\tilde{\alpha}'$-invariant. Therefore, for $\mathtt{Tr}'_{L^2(M,\alpha)\otimes K_r}$ defined as

$$\mathtt{Tr}'_{L^2(M,\alpha)\otimes K_r}(ut'u^*) = \mathtt{Tr}'_{L^2(M)\otimes K}(ut'u^*), \qquad t'\in(\pi(M),\tilde{\alpha})'_+$$

we have

$$\mathtt{Tr}'_H\Big|_{(\pi(M),\tilde{\alpha})'} = \mathtt{Tr}'_{L^2(M,\alpha)\otimes K_r} .$$

If $\mathtt{Tr}'_{H_r}$ denotes $\mathtt{Tr}'_{L^2(M,\alpha)\otimes K_r}$, then we have

$$\mathtt{Tr}'_{H_r} = \mathtt{Tr}'_{H}\Big|_{(\pi(M),\tilde{\alpha})'} .$$

Thus, $\mathtt{Tr}'_{H}\Big|_{(\pi(M),\tilde{\alpha})'}$ is a faithful normal semi-finite trace on $(\pi(M),\tilde{\alpha})'$.

Definition 6.2 *Let M be a finite factor and let α be an involutive *-antiautomorphism of M. Suppose that $\{\pi, H\}$ is a faithful nondegenerate W*-representation of M, and $\pi\circ\alpha = \tilde{\alpha}\circ\pi$ for an involutive *-antiautomorphism $\tilde{\alpha}$ of $\pi(M)$. Then*

$$\dim_{(M,\alpha)}(H_r) = \mathtt{Tr}'_{H_r}(\mathbb{1})$$

is called the **coupling constant** *between $(\pi(M),\tilde{\alpha})$ and $(\pi(M),\tilde{\alpha})'$ relative to H_r.*

(iii) Now, in the case where $\{\pi, H\} = \{\lambda\otimes\mathbb{1}, L^2(M)\otimes K\}$ we choose another basis. Namely, pick a real orthogonal normalized basis $\{f_i\}_{i\in\Lambda'}$ of K, where $|\Lambda'| = \dim_{\mathbb{R}} K$. Then each element $t'\in(\lambda(M)\otimes\mathbb{1}_K)'$ can be uniquely represented as $t' = \big(J\lambda(x_{ij})J\big)$, where $x_{ij}\in M$, for all i,j. If $t'\in(\lambda_r(M,\alpha)\otimes\mathbb{1}_{K_r})'$, i.e. $\tilde{\alpha}'(t') = (t')^*$, then it is not difficult to show that $x_{ij}\in(M,\alpha)$ (i.e. $\alpha(x_{ij}) = x_{ij}^*$), for all i,j. We set

$$\mathtt{tr}'_{L^2(M)\otimes K}(t') = \sum_{i\in\Lambda'}\tau(x_{ii}), \qquad t' = \big(J\lambda(x_{ij})J\big)\in(\lambda(M)\otimes\mathbb{1}_K)'_+ .$$

Clearly, $\mathtt{tr}'_{L^2(M)\otimes K}$ is also a faithful normal semi-finite trace on $(\lambda(M)\otimes\mathbb{1}_K)'_+$. Moreover $\mathtt{tr}'_{L^2(M)\otimes K}$ is $\tilde{\alpha}'$-invariant. Similarly, we can show that the definition of $\mathtt{tr}'_{L^2(M)\otimes K}$ does not depend on the choice of $\{f_i\}$.

In the case where $u\pi(x)u^* = (\lambda(x)\otimes\mathbb{1}_K)p'$ $(x\in M)$ we put

$$\mathtt{tr}'_H(t') = \mathtt{tr}'_{L^2(M)\otimes K}(ut'u^*), \qquad t'\in\pi(M)'_+ .$$

The definition of $\mathtt{tr}'_H$ is also independent on the choice of u and p', and the trace $\mathtt{tr}'_H$ is $\tilde{\alpha}'$-invariant.

Let $\{\pi, H\}$ be a faithful nondegenerate W*-representation of M with $\pi\circ\alpha = \tilde{\alpha}\circ\pi$ for an involutive *-antiautomorphism $\tilde{\alpha}$ of $\pi(M)$.

Definition 6.3 *The number* $\dim_{(M,\alpha)}(H) = \mathtt{tr}'_H(\mathbb{1})$ *is called the* **coupling constant** *between* $(\pi(M), \tilde{\alpha})$ *and* $(\pi(M), \tilde{\alpha})'$ *relative to* H.

One has the following relations between $\dim_{(M,\alpha)}(H_r)$, $\dim_{(M,\alpha)}(H)$ and $\dim_M(H)$.

Theorem 6.4

$$\dim_M(H) = \dim_{(M,\alpha)}(H_r) = \frac{1}{2}\dim_{(M,\alpha)}(H)$$

The proof of this theorem is obvious. $\square$

Let's consider some properties of the coupling constants.

Proposition 6.5 *Let* M $(\subset B(H))$ *be a finite factor and let* α *be an involutive *-antiautomorphism of* M. *Then*

(i) $\dim_{(M,\alpha)}(L^2(M)) = 2$ *and* $\dim_{(M,\alpha)}(L^2(M,\alpha)) = 1$.

(ii) $\dim_{(M,\alpha)}(H) \cdot \dim_{(M,\alpha)'}(H) = 4$ *and* $\dim_{(M,\alpha)}(H_r) \cdot \dim_{(M,\alpha)'}(H_r) = 1$

(iii) *If* $\{\pi, H\}$ *and* $\{\pi', H'\}$ *are* α*-invariant faithful nondegenerate* W**-representations of* M, *then* $\dim_{(M,\alpha)}(H) = \dim_{(M,\alpha)}(H')$ *if and only if* $\{\pi, H\}$ *and* $\{\pi', H'\}$ *are spatially *-isomorphic via a unitary operator* w *with* $\pi(\alpha(w)) = \hat{\alpha}\pi'(w) = \pi'(w)^*$;

(iv) *If* $\{\pi_i, H_i\}_{i\geq 1}$ *is a sequence of* α*-invariant faithful nondegenerate* W**-representations of* M, *then* $\dim_{(M,\alpha)}(\sum\limits_i H_i) = \sum\limits_i \dim_{(M,\alpha)}(H_i)$;

(v) *If* $\{\pi, H\}$ *is an* α*-invariant faithful nondegenerate* W**-representation of* M, *then the following conditions are equivalent:*

a) *real von Neumann algebra* $(\pi(M), \tilde{\alpha})'$ *is finite;*

b) *the trace* Tr'_{H_r} *is finite;*

c) $\dim_{(M,\alpha)}(H) < \infty$.

(vi) $\dim_{(M,\alpha)}(H) \geq 2$ *(resp.* ≤ 2*) if and only if* (M, α) *admits a separating (resp. cyclic) vector.*

Proof. The property (6.1a) and Theorem 6.4 imply the proof of (i). From (6.1b) and Theorem 6.4 we obtain the proof of (ii). The equivalence of the conditions $\dim_{(M,\alpha)}(H) = \dim_{(M,\alpha)}(H')$ and $\dim_M(H) = \dim_M(H')$ follows from Theorem 6.4.

The equivalence of the conditions $\dim_M(H) = \dim_M(H')$ and $\{\pi, H\} \cong \{\pi', H'\}$ follows from (6.1c). By Theorem 5.2 in this case there exists a unitary operator w with $\pi(\alpha(w)) = \hat{\alpha}\pi'(w) = \pi'(w)^*$ which implements this spatial *-isomorphism $\{\pi, H\} \cong \{\pi', H'\}$, what is required to be proved for (iii). From (6.1d) we obtain the proof of (iv). From the theory of real W*-algebras we know that the real von Neumann algebra $(\pi(M), \tilde{\alpha})'$ is finite if and only if the von Neumann algebra $\pi(M)'$ is finite (see [1]). Then by (6.1e) we obtain the proof of (v). It is easy to see that (M, α) admits a separating (respectively, cyclic) vector if and only if M admits a separating (respectively, cyclic) vector. Then from (6.1f) and Theorem 6.4 we obtain the proof of (vi). □

Proposition 6.6 *If $\Re$ is a finite real factor on a real Hilbert space H with the finite commutant $\Re'$, and τ, τ' are the unique faithful normal tracial states on $\Re$ and $\Re'$ respectively, then for any $\xi(\neq 0) \in H$ the number $c_{\Re} = \dfrac{\tau(e_\xi)}{\tau'(e'_\xi)}$ is independent of the choice of ξ. Moreover, we have $c_{\Re} = \dim_{\Re}(H)$, where e_ξ and e'_ξ are the cyclic projections from H onto $\overline{\Re'\xi}$ and $\overline{\Re\xi}$ respectively.*

Proof. We extend τ and τ' to $\Re + i\Re$ and $\Re' + i\Re'$, respectively, by the linearity as $\overline{\tau}(a + ib) = \tau(a) + i\tau(b)$ and $\overline{\tau}'(a' + ib') = \tau'(a') + i\tau'(b')$. It is obvious, that for the cyclic projections $\overline{e}_\xi$ and $\overline{e}'_\xi$ from $H_c = H + iH$ onto $\overline{\Re'\xi} + i\overline{\Re'\xi}$, $\overline{\Re\xi} + i\overline{\Re\xi}$, respectively, we have $\dfrac{\overline{\tau}(\overline{e}_\xi))}{\overline{\tau}'(\overline{e}'_\xi)} = \dfrac{\tau(e_\xi)}{\tau'(e'_\xi)}$. Since $\dfrac{\overline{\tau}(\overline{e}_\xi))}{\overline{\tau}'(\overline{e}'_\xi)}$ is independent on the choice of ξ, and this number is equal to $\dim_{\Re+i\Re}(H + iH)$, we obtain that $\dim_{\Re}(H) = c_{\Re}$. □

Example 6.7 Let M be a factor of type I_n on an m-dimensional Hilbert space H $(n, m < \infty)$, and let α be an involutive *-antiautomorphism of M. It is known that $\dim_M(H) = m/n^2$, i.e $\dim_M(H) = \dim(H)/\dim(M)$. Then by Theorem 6.4 we have $\dim_{(M,\alpha)}(H) = 2m/n^2$, hence, since $\dim(M, \alpha) = n^2$, one has $\dim_{(M,\alpha)}(H) = \frac{2\dim(H)}{\dim(M)} = \dim_{\mathbb{R}}(H)/\dim(M, \alpha)$.

From the theory of real W*-algebras we know (see [1,4,17]), that if n is an odd number, then M possesses exactly one conjugacy class of involutive *-antiautomorphism, and if n is an even number, then there are two conjugacy class of involutive *-antiautomorphism

of M. Namely, in the first case $(M,\alpha) \cong M_n(\mathbb{R})$, and in the second case we have also the possibility $(M,\alpha) \cong M_{n/2}(\mathbb{H})$, where $\mathbb{H}$ is the quaternion algebra. Thus

$$\dim_{M_n(\mathbb{R})}(H) = \frac{\dim_{\mathbb{R}}(H)}{n^2} = \frac{\dim_{\mathbb{R}}(H)}{\dim(M_n(\mathbb{R}))} ,$$

$$\dim_{M_{n/2}(\mathbb{H})}(H) = \frac{4\dim_{\mathbb{H}}(H)}{n^2} = \frac{\dim_{\mathbb{H}}(H)}{\left(\frac{n}{2}\right)^2} = \frac{\dim_{\mathbb{H}}(H)}{\dim_{\mathbb{H}}(M_{n/2}(\mathbb{H}))} .$$

7 The index of subfactors in finite real factors

Definition 7.1 *Let $M \subset B(H)$ be a finite factor. Consider a subfactor $N \subset M$ and let α be an involutive *-antiautomorphism of M with $\alpha(N) \subset N$. Consider the real factors $\Re = (M,\alpha)$ and $Q = (N,\alpha)$. The* **index** *of Q in $\Re$, denoted by $[\Re : Q]$ or by $[(M,\alpha) : (N,\alpha)]$, is defined as $\dim_Q(L^2(\Re))$.*

Between real and complex indices there is the following relation

Theorem 7.2 *We have $[(M,\alpha) : (N,\alpha)] = [M,N]$, i.e. $[\Re : Q] = [\Re + i\Re : Q + iQ]$.*

Proof. By Theorem 6.4 we have $[(M,\alpha) : (N,\alpha)] = \dim_{(N,\alpha)}(L^2(M,\alpha)) = \frac{1}{2}\dim_{(N,\alpha)}(L^2(M)) = \frac{1}{2}\cdot 2\dim_N(L^2(M)) = [M,N]$ $\square$

We recall some properties of the complex index ([6], [8], [10, Ch. 17]): if M is a finite factor and N is a subfactor of M then

(7.1a) $[M : N] = \dim_N(H)/\dim_M(H)$,

(7.1b) $[M : N] \geq 1$, in particular, $[M : M] = 1$,

(7.1c) $[M : N] = [N' : M']$,

(7.1d) If P is a subfactor of N, then $[M : P] = [M : N]\cdot[N : P]$,

(7.1e) If P is a subfactor of N, $[M : P] < \infty$ and $[M : P] = [M : N]$, then $N = P$,

(7.1f) If M_i is a finite factor and N_i is a subfactor of $M_i (i = 1, 2)$ then $[M_1 \overline{\otimes} M_2 : N_1 \overline{\otimes} N_2] = [M_1 : N_1]\cdot[M_2 : N_2]$

Similarly to the complex case, using Theorem 7.2 we can prove the following properties of the real index

Theorem 7.3 *Let H_r be a real Hilbert space. Suppose that $\Re \subset B(H_r)$ is a finite real factor and $Q \subset \Re$ is a real subfactor. Let $M = \Re + i\Re$ be the enveloping complex factor for $\Re$ and let α be the involutive *-antiautomorphism of M which generates $\Re$, i.e. $\Re = (M, \alpha)$ (in this case it is clear that $Q = (N, \alpha)$, where $N = Q + iQ$). Then*

(i) $[(M,\alpha):(N,\alpha)] = \dfrac{\dim_{(N,\alpha)}(H_r)}{\dim_{(M,\alpha)}(H_r)}$, *i.e.* $[\Re : Q] = \dfrac{\dim_Q(H_r)}{\dim_\Re(H_r)}$.

(ii) $[(M,\alpha):(N,\alpha)] \geq 1$, *i.e.* $[\Re : Q] \geq 1$. *In particular,* $[(M,\alpha):(M,\alpha)] = [\Re : \Re] = 1$.

(iii) $[(M,\alpha):(N,\alpha)] = [(N,\alpha)' : (M,\alpha)']$, *i.e.* $[\Re : Q] = [Q' : \Re']$.

(iv) *If Q_1 is a real subfactor of Q, then* $[(M,\alpha):(N_1,\alpha)] = [(M,\alpha):(N,\alpha)] \cdot [(N,\alpha):(N_1,\alpha)]$, *i.e.* $[\Re : Q_1] = [\Re : Q] \cdot [Q : Q_1]$, *where $N_1 = Q_1 + iQ_1$.*

(v) *If Q_1 is a real subfactor of Q, $[\Re : Q_1] < \infty$ and $[(M,\alpha):(N_1,\alpha)] = [(M,\alpha):(N,\alpha)]$, then $N = N_1$, i.e. if $[\Re : Q_1] = [\Re : Q]$, then $Q = Q_1$, where $N_1 = Q_1 + iQ_1$.*

(vi) *Let $\Re_i$ be a finite real factor, and let Q_i be a real subfactor of $\Re_i$, $i = 1, 2$. If $M_i = \Re_i + i\Re_i$ and $N_i = Q_1 + iQ_i$ are the enveloping complex factors for $\Re_i$ and Q_i, respectively, then let α_i denote the involutive *-antiautomorphism of M_i, which generates $\Re_i$, i.e. $\Re_i = (M_i, \alpha_i)$, $i = 1, 2$. Then*

$[(M_1,\alpha_1)\overline{\otimes}(M_2,\alpha_2) : (N_1,\alpha_1)\overline{\otimes}(N_2,\alpha_2)] = [(M_1,\alpha_1) : (N_1,\alpha_1)] \cdot [(M_2,\alpha_2) : (N_2,\alpha_2)]$, *i.e.* $[\Re_1\overline{\otimes}\Re_2 : Q_1\overline{\otimes}Q_2] = [(\Re_1 : Q_1] \cdot [\Re_2 : Q_2]$.

Proof. (i). By Theorem 7.2 and the property (7.1a) we have $[(M,\alpha):(N,\alpha)] =$

$$= [M : N] = \frac{\dim_N(H_r + iH_r)}{\dim_M(H_r + iH_r)} = \frac{\frac{1}{2}\dim_{(N,\alpha)}(H_r + iH_r)}{\frac{1}{2}\dim_{(M,\alpha)}(H_r + iH_r)} = \frac{\dim_{(N,\alpha)}(H_r)}{\dim_{(M,\alpha)}(H_r)}.$$

(ii). By the Theorem 7.2 and (7.1b) we have $[(M,\alpha):(N,\alpha)] = [M : N] \geq 1$ and

$[(M,\alpha):(M,\alpha)] = [M:M] = 1.$

(iii). As above, let α' be the involutive *-antiautomorphism of M', which generates $\Re'$, i.e. $\Re' = (M',\alpha')$. Then $(M,\alpha)' = (M',\alpha')$. Similarly we have $(N,\alpha)' = (N',\alpha')$. Hence by Theorem 7.2 and the property (7.1c) we have $[(M,\alpha):(N,\alpha)] = [M:N] = [N':M'] = [(N',\alpha'):(M',\alpha')] = [(N,\alpha)':(M,\alpha)']$.

(iv). By Theorem 7.2 and the property (7.1d) we have $[(M,\alpha):(N_1,\alpha)] = [M:N_1] = [M:N]\cdot[N:N_1] = [(M,\alpha):(N,\alpha)]\cdot[(N,\alpha):(N_1,\alpha)]$.

(v). If $[(M,\alpha):(N_1,\alpha)] = [(M,\alpha):(N,\alpha)]$, by Theorem 7.2 we have $[M:N_1] = [(M,\alpha):(N_1,\alpha)] = [(M,\alpha):(N,\alpha)] = [M:N]$, i.e. $[M:N_1] = [M:N]$. Then by (7.1e) we obtain $N = N_1$, i.e. $Q = Q_1$.

(vi). According to [11, page 69] we have $\Re_1\overline{\otimes}\Re_2 + i\Re_1\overline{\otimes}\Re_2 = (\Re_1 + i\Re_1)\overline{\otimes}(\Re_2 + i\Re_2)$, i.e. $(M_1,\alpha_1)\overline{\otimes}(M_2,\alpha_2) + i(M_1,\alpha_1)\overline{\otimes}(M_2,\alpha_2) = M_1\overline{\otimes}M_2$. Similarly, we have $(N_1,\alpha_1)\overline{\otimes}(N_2,\alpha_2) + i(N_1,\alpha_1)\overline{\otimes}(N_2,\alpha_2) = N_1\overline{\otimes}N_2$. Then by Theorem 7.2 and the property (7.1f) we get

$[(M_1,\alpha_1)\overline{\otimes}(M_2,\alpha_2) : (N_1,\alpha_1)\overline{\otimes}(N_2,\alpha_2)] =$

$[(M_1,\alpha_1)\overline{\otimes}(M_2,\alpha_2) + i(M_1,\alpha_1)\overline{\otimes}(M_2,\alpha_2) \; : \; (N_1,\alpha_1)\overline{\otimes}(N_2,\alpha_2) +$

$i(N_1,\alpha_1)\overline{\otimes}(N_2,\alpha_2)] = [((M_1,\alpha_1) + i(M_1,\alpha_1))\overline{\otimes}((M_2,\alpha_2) + i(M_2,\alpha_2)) :$

$((N_1,\alpha_1) + i(N_1,\alpha_1))\overline{\otimes}((N_2,\alpha_2) + i(N_2,\alpha_2))] = [M_1\overline{\otimes}M_2 : N_1\overline{\otimes}N_2] =$

$[M_1:N_1]\cdot[M_2:N_2] = [(M_1,\alpha_1):(N_1,\alpha_1)]\cdot[(M_2,\alpha_2):(N_2,\alpha_2)]$.

Thus, we have $[\Re_1\overline{\otimes}\Re_2 : Q_1\overline{\otimes}Q_2] = [(\Re_1 : Q_1]\cdot[\Re_2 : Q_2]$. $\square$

As it was noted in the introduction, V.Jones in [6] has proved a theorem on the values of index for subfactors. Let us recall this theorem

Theorem 7.4 ([6], Theorem 4.3.1) *Let M be a finite factor, and let N be a subfactor of M with $[M : N] < \infty$. Then one has either $[M : N] = 4\cos^2 \dfrac{\pi}{q}$ for some integer $q \geq 3$ or $[M : N] \geq 4$.*

From Theorems 7.2 and 7.4 we obtain the following real version of the above theorem.

Theorem 7.5 *Let M be a finite factor and let N be a subfactor of M with $[M : N] < \infty$. Given be an involutive *-antiautomorphism α of M with $\alpha(N) \subset N$, put $\Re = (M, \alpha)$, $Q = (N, \alpha)$. Then one has either $[(M, \alpha) : (N, \alpha)] = 4\cos^2 \dfrac{\pi}{q}$ for some integer $q \geq 3$ or $[(M, \alpha) : (N, \alpha)] \geq 4$, i.e. $[\Re : Q] = 4\cos^2 \dfrac{\pi}{q}$ for some integer $q \geq 3$ or $[\Re : Q] \geq 4$.*

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